

ON THE LAW OF THE ITERATED LOGARITHM FOR BROWNIAN MOTION ON COMPACT MANIFOLDS

CHENG OUYANG AND JENNIFER PAJDA-DE LA O

ABSTRACT. By taking a functional analytic point of view, we consider a family of distributions (continuous linear functionals on smooth functions), denoted by $\{\mu_t, t > 0\}$, associated to the law of iterated logarithm for Brownian motion on a compact manifold. We give a complete characterization of the collection of limiting distributions of $\{\mu_t, t > 0\}$.

1. INTRODUCTION

Let M be a compact C^∞ -Riemannian manifold (without boundary). For any fixed $x \in M$, it is well-known that the Laplace-Beltrami operator Δ_M generates a unique diffusion process X starting from x , which is called the Brownian motion on M starting from x . It is a continuous, strong Markov process with transition density $p(t, x, y)$, the fundamental solution of

$$\frac{\partial}{\partial t} p(t, x, \cdot) = \frac{1}{2} \Delta_M p(t, x, \cdot).$$

Denote by m the volume measure on M induced by the metric and $m_0 = m(M)$. Since m/m_0 is the invariant probability measure for X , the well-known ergodic theorem implies that for all $f \in L^1(M)$, almost surely

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(X_s) ds = \frac{1}{m_0} \int_M f dm.$$

Hence, $\int_0^t f(X_s) ds$ blows up to infinity with a rate of t and scalar $\frac{1}{m_0} \int_M f dm$. The second order term is given by

$$\int_0^t f(X_s) ds - \frac{t}{m_0} \int_M f dm.$$

When t tends to infinity, the magnitude of the above is characterized by the law of the iterated logarithm (see, for example [1] and [2]). More precisely, we have almost surely

$$(1.1) \quad \limsup_{t \rightarrow \infty} \frac{\int_0^t f(X_s) ds - m_0^{-1} t \int_M f dm}{\sqrt{2t \log \log t}} = \sqrt{\frac{2}{m_0}} (Gf, f)_{L^2}.$$

In the above, $(\cdot, \cdot)_{L^2}$ is the standard inner product associated to $L^2(M)$, and G is the Green operator introduced by Baxter and Brosamler in [1] (see Section 2 below for an explicit definition of G). It took some effort to show that (1.1) is true simultaneously for all $f \in C^\infty(M)$. Indeed, the following was proved by Brosamler in [2].

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Theorem 1.1. *We have*

$$(1.2) \quad \mathbb{P} \left\{ \limsup_{t \rightarrow \infty} \frac{\int_0^t f(X_s) ds - m_0^{-1} t \int_M f dm}{\sqrt{2t \log \log t}} = \sqrt{\frac{2}{m_0}} (Gf, f)_{L^2}, \text{ all } f \in C^\infty \right\} = 1.$$

The present work is concerned with the family of signed measures $\{\mu_t, t > 0\}$ on M obtained by

$$\int_M f d\mu_t = \frac{\int_0^t f(X_s) ds - m_0^{-1} t \int_M f dm}{\sqrt{2t \log \log t}}.$$

With Theorem 1.1 in mind, we would rather think of μ_t as a distribution (continuous linear functional) on $C^\infty(M)$ and write, for all $f \in C^\infty(M)$

$$\mu_t(f) = \int_M f d\mu_t.$$

Of course, μ_t also depends on the sample path $\omega \in \Omega$ of X . In the event that we want to emphasize such dependence, we will write μ_t^ω . Throughout this paper, we use the term *distributions* exclusively for continuous linear functionals on $C^\infty(M)$. It is then natural to wonder what the limiting distributions of the family $\{\mu_t\}$ are. More precisely, we are interested in, for each $\omega \in \Omega$, how one can characterize the class of distributions μ on $C^\infty(M)$ such that there exists a sequence $\{t_n, n \geq 1\}$

$$(1.3) \quad \mu_{t_n}^\omega(f) \rightarrow \mu(f), \quad \text{all } f \in C^\infty(M).$$

The result of our investigation on the above question is reported in the next theorem.

Theorem 1.2. *Let \mathcal{D} be the collection of distributions on $C^\infty(M)$ satisfying the following four properties.*

- (a) μ can be identified as a signed measure on M , still denoted by μ ;
- (b) $\mu(M) = 0$;
- (c) μ is absolutely continuous with respect to the volume measure m , with Radon-Nikodym derivative $g = d\mu/dm$ in $L^2(M)$. Moreover,
- (d) g is in the domain of $G_{1/2}^{-1}$, and

$$\|G_{\frac{1}{2}}^{-1}g\|_{L^2} \leq \sqrt{\frac{2}{m_0}}.$$

Here, roughly speaking, $G_{1/2} = (-\Delta_M/2)^{-1/2}$, and is defined more precisely in the next section.

Then, almost surely, the class of limiting distributions of the family $\{\mu_t^\omega, t > 0\}$ is exactly \mathcal{D} .

Clearly, the above theorem gives a complete characterization of the collection of limiting distributions of $\{\mu_t\}$.

Careful readers may wonder whether Theorem 1.2 remains valid if we replace $C^\infty(M)$ in (1.3) by $C(M)$, the collection of (bounded) continuous functions on M . That is, we regard $\{\mu_t\}$ and μ in Theorem 1.2 as genuine signed measures and consider weak convergence in the space of signed measures, as opposed to convergence in the space of distributions. The main reason why we do not work in

the former setting in the present work is due to the fact that, in that case, Theorem 1.2 is intimately related to a version of the law of the iterated logarithm that holds simultaneously for all $f \in C(M)$. Whether one has such an iterated logarithm is a non-trivial question. It seems the best result in this direction is obtained by Brosamler in [2] for the Sobolev spaces H_0^α with $\alpha > \max(d - 3/2, d/2)$, where d is the dimension of M and H_0^α is that in Definition 2.1 below.

The rest of this paper has two sections. In Section 2, we provide some preliminary material that will be needed for our discussion later. In particular, we introduce some operators and Sobolev spaces associated to Brownian motion on manifolds. The proof of Theorem 1.2 is detailed in Section 3.

2. BROWNIAN MOTION ON MANIFOLDS AND SOBOLEV SPACES

Throughout our discussion below, we assume that M is a compact Riemannian manifold of dimension d . Fix any $x \in M$, let $X = \{X_t, t \geq 0\}$ be a Brownian motion on M starting from x ; that is, X is the unique diffusion process generated by the Laplace-Beltrami operator Δ_M . Denote by $p(t, x, y)$ its probability transition density function. In this section, we briefly introduce some operators and Sobolev spaces associated to X , that will be needed in the sequel. A more detailed discussion on these materials can be found, e.g., in [2].

We first introduce the Green kernel

$$(2.1) \quad g(x, y) = \int_0^\infty (p(t, x, y) - m_0^{-1}) dt, \quad x, y \in M, x \neq y.$$

Clearly $g(x, y)$ inherits its symmetry in x and y from $p(t, x, y)$. It is also not hard to see that g is continuous off the diagonal of $M \times M$. Since for large t , there exists $\alpha > 0$ and $C > 0$ such that (see, e.g., [1])

$$\sup_{x, y \in M} |p(t, x, y) - m_0^{-1}| \leq C e^{-\alpha t},$$

and for small t , $p(t, x, y)$ is known to have an order (see, e.g., [4] and [3])

$$(2\pi t)^{-\frac{d}{2}} e^{-\frac{d(x, y)^2}{2t}}.$$

Here $d(x, y)$ is the Riemannian distance between x and y . Moreover, with some extra work, one can show that $g(x, y)$ is C^∞ off the diagonal.

More generally, we introduce for $\alpha > 0$ the kernel

$$(2.2) \quad g_\alpha(x, y) = \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1} (p(t, x, y) - m_0^{-1}) dt, \quad x, y \in M, x \neq y.$$

Obviously $g_1(x, y) = g(x, y)$, and $g_\alpha(x, y)$ is symmetric in x and y . The semigroup property of $p(t, x, y)$ implies that

$$\int_M g_\alpha(x, z) g_\beta(z, y) m(dz) = g_{\alpha+\beta}(x, y), \quad \text{for } \alpha, \beta > 0.$$

Hence, for any $f \in L^2(M)$, letting

$$(G_\alpha f)(x) = \int_M g_\alpha(x, y) f(y) m(dy),$$

we obtain a semigroup of bounded symmetric linear operators G_α on $L^2(M)$. In particular, we denote $G = G_1$ which is the Green operator introduced in [1].

In the following, we let

$$L_0^2(M) = \left\{ f \in L^2(M) : \int_M f dm = 0 \right\},$$

and

$$C_0^\infty(M) = \left\{ f \in C^\infty(M) : \int_M f dm = 0 \right\}.$$

For simplicity, we will suppress M in the notation when there is no danger of possible confusion.

Definition 2.1. For any $\alpha > 0$, let $H_0^\alpha = G_{\alpha/2}(L_0^2)$, with inner product

$$(G_{\frac{\alpha}{2}} f_1, G_{\frac{\alpha}{2}} f_2)_{H_0^\alpha} = 2^\alpha (f_1, f_2)_{L^2}.$$

We write $\|\cdot\|_{H_0^\alpha}$ for the norm induced by $(\cdot, \cdot)_{H_0^\alpha}$.

It is known (see, e.g., [2]) that for $\alpha_1 < \alpha_2$, $H_0^{\alpha_2}$ is continuously embedded into $H_0^{\alpha_1}$, and $\cap_{\alpha>0} H_0^\alpha = C_0^\infty$. Moreover, the Sobolev spaces H_0^α are the completion of C_0^∞ with respect to the norm $\|\cdot\|_{H_0^\alpha}$.

The characterization of operators G_α and spaces H_0^α is probably more familiar to some readers in a functional analytic setting. Denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ the eigenvalues of $-\Delta_M$ and by $\phi_0 = m_0^{-1/2}, \phi_1, \phi_2, \dots$ an orthonormal sequence of corresponding eigenfunctions.

Proposition 2.2. *The following facts are well-known.*

- (1) $\phi_n \in C^\infty$, $n \geq 0$.
- (2) $G_\alpha \phi_0 = 0$ and

$$G_\alpha \phi_n = 2^\alpha \lambda_n^{-\alpha} \phi_n,$$

for $\alpha > 0$.

- (3) Set

$$\phi_n^\alpha = \lambda_n^{-\alpha/2} \phi_n.$$

For all $\alpha > 0$, the functions $\{\phi_n^\alpha, n \geq 1\}$ form a complete orthonormal system in H_0^α .

- (4) For any $f \in L_0^2$, let $f_n = (f, \phi_n)_{L^2}$. We have

$$f = \sum_{n=1}^{\infty} f_n \phi_n.$$

Moreover, f belongs to H_0^α if and only if

$$\sum_{n=1}^{\infty} \lambda_n^\alpha f_n^2 < \infty.$$

- (5) For $f \in H_0^\alpha$,

$$\|f\|_{H_0^\alpha}^2 = \sum_{n=1}^{\infty} \lambda_n^\alpha f_n^2.$$

Remark 2.3. By the characterization in terms of eigenvalues and eigenfunctions, it is clear that $G_\alpha : L_0^2 \rightarrow H_0^{2\alpha}$ is a self-adjoint operator and, indeed, $G_\alpha = (-\Delta_M/2)^{-\alpha}$.

Finally, we state one of the main results in [2], which plays a key role in our discussion below.

Theorem 2.4. *For $\alpha > \max(d - 3/2, d/2)$, we have almost surely*

$$\left| \frac{\int_0^t f(X_s) ds}{\sqrt{2t \log \log t}} \right| \leq \|f\|_{H_0^\alpha} C(\omega), \quad t \geq 3, f \in H_0^\alpha.$$

In the above, C is a finite constant depending on sample path ω (but not on t).

Proof. This is essentially the content of [2, Theorem 3.8]. \square

3. LIMITING DISTRIBUTIONS

Recall that X is a Brownian motion on a compact manifold M starting from a pre-fixed point $x \in M$. For each $t > 0$ and $\omega \in \Omega$, we consider the distribution μ_t^ω on $C^\infty(M)$ obtained by

$$\mu_t^\omega(f) = \frac{\int_0^t f(X_s(\omega)) ds - m_0^{-1} t \int_M f dm}{\sqrt{2t \log \log t}}, \quad f \in C^\infty(M).$$

To lighten the notation, we usually suppress its dependence on ω and simply write μ_t . We are interested in understanding the class of limiting distributions (accumulating points) of the family $\{\mu_t\}$. Clearly, in order to prove Theorem 1.2, we only need to show the following two theorems hold.

Theorem 3.1. *For each $\omega \in \Omega$, if μ is a limiting distribution of the family $\{\mu_t^\omega, t \geq 0\}$, then μ can be identified as a signed measure on M , still denoted by μ , such that*

- (a) $\mu(M) = 0$;
- (b) μ is absolutely continuous with respect to the volume measure m , with Radon-Nikodym derivative $g = d\mu/dm$ in $L_0^2(M)$. Moreover,
- (c) g is in the domain of $G_{1/2}^{-1}$, and

$$\left\| G_{\frac{1}{2}}^{-1} g \right\|_{L^2} \leq \sqrt{\frac{2}{m_0}}.$$

Theorem 3.2. *There exists a subset $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for any signed measure μ satisfying the characterizations in Theorem 3.1 and any $\omega \in \Omega_0$, we can find a sequence of times $\{t_n, n \geq 1\}$ such that for all $f \in C^\infty(M)$,*

$$\mu_{t_n}^\omega(f) \rightarrow \mu^\omega(f),$$

as $n \rightarrow \infty$.

The rest of this section is devoted to the proof of Theorem 3.1 and Theorem 3.2 above.

Proof of Theorem 3.1. Fix any $\omega \in \Omega$ in (1.2), and suppose μ is a limiting distribution of the family $\{\mu_t^\omega, t \geq 0\}$. First note that μ can be identified as (or, in another word, extended to) a signed measure on M . Indeed, by (1.2) and the fact that $G : L^2(M) \rightarrow L^2(M)$ is a bounded linear operator, we have for all $f \in C^\infty$,

$$|\mu(f)| \leq \sqrt{\frac{2}{m_0}} (f, Gf)_{L^2} \leq C \sqrt{\frac{2}{m_0}} \|f\|_{L^2}$$

for some constant $C > 0$. Since $C^\infty(M)$ is dense in $L^2(M)$, the above inequality implies that μ can be extended to (and hence be identified as) a bounded linear

functional on $L^2(M)$. Now, the Riesz representation theorem tells us that there exists a function $g \in L^2(M)$ such that

$$\mu(f) = (g, f)_{L^2} = \int_M f g dm, \quad f \in L^2(M).$$

As a consequence, we can identify $\mu = g dm$, a signed measure on M which is absolutely continuous with respect to the volume measure m . Clearly, the Radon-Nikodym derivative g is $L^2(M)$. In addition, for any constant function f , we have

$$\mu_t(f) = 0.$$

It implies

$$\mu(f) = 0,$$

and in particular for $f \equiv 1$,

$$\mu(M) = \int_M g dm = 0.$$

Hence $g \in L_0^2(M)$. This proves (a) and (b) of Theorem 3.1.

Next, we show that g satisfies (c) of Theorem 3.1. For any $f \in C_0^\infty$, denote by

$$h = G_{\frac{1}{2}} f.$$

By (1.2), we have

$$\begin{aligned} \left| \int_M f g dm \right| &= |\mu(f)| \\ &\leq \sqrt{\frac{2}{m_0}} (f, Gf)_{L^2} \\ &= \sqrt{\frac{2}{m_0}} (G_{\frac{1}{2}} f, G_{\frac{1}{2}} f)_{L^2} \\ &= \sqrt{\frac{2}{m_0}} \|h\|_{L^2}. \end{aligned}$$

That is

$$(3.1) \quad \left| (G_{\frac{1}{2}}^{-1} h, g)_{L^2} \right| = \left| \int_M \left(G_{\frac{1}{2}}^{-1} h \right) g dm \right| \leq \sqrt{\frac{2}{m_0}} \|h\|_{L^2},$$

for all $h = G_{1/2} f$, $f \in C_0^\infty$. Observe that $C_0^\infty = \cap_{\alpha \geq 0} H_0^\alpha$, together with the definition of H_0^α , we have $G_{1/2}(C_0^\infty) = C_0^\infty$. Thus we conclude that (3.1) holds valid for all $h \in C_0^\infty$. As a consequence, g is in the domain of $(G_{1/2}^{-1})^*$, the adjoint of $G_{1/2}^{-1}$, for C_0^∞ is dense in L_0^2 . Thus

$$\left(h, \left(G_{\frac{1}{2}}^{-1} \right)^* g \right)_{L^2} \leq \sqrt{\frac{2}{m_0}} \|h\|_{L^2}.$$

Again, the density of C_0^∞ in L_0^2 and the above inequality implies

$$\left\| \left(G_{\frac{1}{2}}^{-1} \right)^* g \right\|_{L^2} \leq \sqrt{\frac{2}{m_0}}.$$

Now the proof of (c) is completed by observing that $G_{1/2}^{-1}$ is a self-adjoint operator on L_0^2 . \square

Finally, we focus on the proof of Theorem 3.2. First, observe that for $f_1, \dots, f_n \in L_0^2$, the matrix $((f_i, Gf_j)_{L^2}, i, j = 1, \dots, n)$ is positive definite if and only if f_1, \dots, f_n are linearly independent. Let $f_1, \dots, f_n \in L_0^2$ be linearly independent and consider the ellipsoid E_{f_1, \dots, f_n} defined by

$$(3.2) \quad E_{f_1, \dots, f_n} = \left\{ (z_1, \dots, z_n) \in \mathbb{R}^n, \sum_{i,j=1}^n a_{ij} z_i z_j \leq 1 \right\}.$$

Here $(\frac{m_0}{2} a_{ij})$ is the inverse matrix of $((f_i, Gf_j)_{L^2}, i, j = 1, \dots, n)$.

Remark 3.3. Recall our ϕ_1, ϕ_2, \dots in Proposition 2.2. Clearly ϕ_1, ϕ_2, \dots are linearly independent and $\phi_k \in C_0^\infty$ for all $k \geq 1$. Since

$$G\phi_k = 2\lambda_k^{-1} \phi_k,$$

we have for $f_k = \sqrt{\lambda_k/2} \phi_k$,

$$(f_i, Gf_j)_{L^2} = \delta_{ij}.$$

Throughout our discussion below we pick this particular choice of f_k 's. In this case, for each $n \geq 1$, E_{f_1, \dots, f_n} is simply a ball in \mathbb{R}^n centered at the origin with radius $\sqrt{2/m_0}$.

Lemma 3.4. *Suppose $\alpha > \max(d - \frac{3}{2}, \frac{d}{2})$, and denote by*

$$L_t(f) = \int_0^t f(X_s) ds.$$

There exists a subset $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $n \geq 1$

$$\mathbb{R}^n - \text{cluster set } \frac{(L_t(f_1), \dots, L_t(f_n))}{\sqrt{2t \log \log t}} = E_{f_1, \dots, f_n}, \quad \text{when } t \rightarrow \infty.$$

Proof. This is a restatement of Theorem 4.6 of [2]. □

Proof of Theorem 3.2. We want to show that for any $\omega \in \Omega_0$ in Lemma 3.4 and any fixed μ , a signed measure satisfying the characterizations in Theorem 3.1, we can find a sequence of times $t_1 < t_2 < t_3 < \dots$ such that

$$\mu_{t_n} \rightarrow \mu, \quad \text{as } n \rightarrow \infty,$$

in the space of distributions. We break our proof into three steps.

Step 1. We first show that for any $n \geq 1$, the vector

$$v_n = (\mu(f_1), \dots, \mu(f_n))$$

is an element in the ball E_{f_1, \dots, f_n} defined in (3.2).

By our choice of $\{f_k, k \geq 1\}$ in Remark 3.3, E_{f_1, \dots, f_n} is a ball in \mathbb{R}^n centered at the origin with radius $\sqrt{2/m_0}$. Hence the proof reduces to show that the inner product $v_n \cdot \zeta$ in \mathbb{R}^n satisfies

$$(3.3) \quad |v_n \cdot \zeta| \leq \sqrt{\frac{2}{m_0}},$$

for any unit vector $\zeta = (\zeta^1, \dots, \zeta^n) \in \mathbb{R}^n$. Indeed, we have

$$\begin{aligned}
v_n \cdot \zeta &= \sum_{k=1}^n \mu(f_k) \zeta^k \\
&= \mu \left(\sum_{k=1}^n \zeta^k f_k \right) \\
&= \int_M \left(\sum_{k=1}^n \zeta^k f_k \right) g \, dm \\
&= \left(\sum_{k=1}^n \zeta^k f_k, g \right)_{L^2} \\
&= \left(\sum_{k=1}^n \zeta^k G_{\frac{1}{2}} f_k, G_{\frac{1}{2}}^{-1} g \right)_{L^2},
\end{aligned}$$

where $g = d\mu/dm$. Hence, by the Cauchy-Schwarz inequality and our choice of f_k and μ , we obtain

$$\begin{aligned}
|v_n \cdot \zeta| &\leq \left\| \sum_{k=1}^n \zeta^k G_{\frac{1}{2}} f_k \right\|_{L^2} \left\| G_{\frac{1}{2}}^{-1} g \right\|_{L^2} \\
&= \sqrt{(\zeta^1)^2 + \dots + (\zeta^n)^2} \left\| G_{\frac{1}{2}}^{-1} g \right\|_{L^2} \\
&\leq \sqrt{\frac{2}{m_0}},
\end{aligned}$$

where we used the fact that

$$\left\| G_{\frac{1}{2}}^{-1} g \right\|_{L^2} \leq \sqrt{\frac{2}{m_0}}.$$

Hence we have proved the desired inequality in (3.3).

Step 2. Fix any $\omega \in \Omega_0$ in Lemma 3.4. We show in this step that we can find a sequence of times $t_1 < t_2 < t_3 < \dots$ such that

$$(3.4) \quad \mu_{t_n}(f_k) \rightarrow \mu(f_k), \quad \text{as } n \rightarrow \infty,$$

for all $k \geq 1$.

For each fixed $n \geq 1$, still let

$$v_n = (\mu(f_1), \dots, \mu(f_n)).$$

Recall that for $f \in C_0^\infty$,

$$\mu_t(f) = \frac{\int_0^t f(X_s) ds}{\sqrt{2t \log \log t}} = \frac{L_t(f)}{\sqrt{2t \log \log t}}.$$

Introduce

$$v_{n,t} = (\mu_t(f_1), \dots, \mu_t(f_n)).$$

By Lemma 3.4 and what we have proved in *Step 1*, for each fixed n there exists an increasing sequence of times $\{t_m^n, m \geq 1\}$ such that

$$|v_{n,t_m^n} - v_n| \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

Start with $n = 1$. Because $v_{1,t_m^1} \rightarrow v_1$, in particular, we can find $t_1 \in \{t_m^1\}$ such that

$$|v_{1,t_1} - v_1| < 1.$$

For $n = 2$, $v_{2,t_m^2} \rightarrow v_2$ implies that we can find $t_2 \in \{t_m^2\}$ such that $t_2 > t_1$ and

$$|v_{2,t_2} - v_2| < \frac{1}{2}.$$

In general, we can choose $t_n \in \{t_m^n\}$ such that $t_n > t_{n-1}$ and

$$(3.5) \quad |v_{n,t_n} - v_n| < \frac{1}{n}.$$

We claim that with such choice of $\{t_n, n \geq 1\}$, the convergence in (3.4) holds true for all k . Indeed, for each fixed k , we observe that $\mu_{t_n}(f_k) - \mu(f_k)$ is the k -th entry of the vector $v_{n,t_n} - v_n$ when $n \geq k$. Hence by (3.5),

$$|\mu_{t_n}(f_k) - \mu(f_k)| \leq |v_{n,t_n} - v_n| \leq \frac{1}{n}, \quad n \geq k.$$

Letting $n \rightarrow \infty$, the proof is completed.

We emphasize that our choice of $\{t_n\}$ in this step may depend on $\omega \in \Omega_0$.

Step 3. We complete our proof of Theorem 3.2 in this step. That is, we show for any fixed $\omega \in \Omega_0$ in Lemma 3.4 and any μ in Theorem 3.1, there exists a sequence of times $\{t_n, n \geq 1\}$ (that may depend on ω) such that for all $f \in C^\infty$,

$$\mu_{t_n}(f) \rightarrow \mu(f).$$

Obviously, since $\mu_t(f) = \mu(f) = 0$ for constant f , we only need to prove the desired convergence for all $f \in C_0^\infty$.

Let

$$\mathcal{L} = \{f, f \text{ is a (finite) linear combination of } f_1, f_2, \dots\}.$$

Clearly \mathcal{L} is a dense subset of H_0^α for all $\alpha > 0$, for $\{\phi_n^\alpha, n \geq 1\}$ is a complete orthonormal system in H_0^α .

On the other hand, by what we have proved in *Step 2*, together with the linearity of both $\mu_t(\cdot)$ and $\mu(\cdot)$, for each fixed $\omega \in \Omega_0$ there exists $\{t_n, n \geq 1\}$ such that for all $f \in \mathcal{L}$,

$$\mu_{t_n}(f) \rightarrow \mu(f).$$

By a standard density argument, in order to conclude our proof it suffices to show that, uniformly in t , $\mu_t(\cdot)$ is a continuous functional on the space H_0^α for some $\alpha > 0$, and that $\mu(\cdot)$ is a continuous functional on the same H_0^α . Fortunately, the desired uniform continuity of $\mu_t(\cdot)$ is given by Theorem 2.4 for any $\alpha > \max(d - 3/2, d/2)$.

For the continuity of $\mu(\cdot)$, we only need to note that for $f \in H_0^\alpha$,

$$\begin{aligned} |\mu(f)| &= |(f, g)_{L^2}| \\ &\leq C_1 \|f\|_{L^2} \|g\|_{L^2} \\ &\leq C_2 \|f\|_{H_0^\alpha} \|g\|_{H_0^1} \\ &\leq 2C_2 \sqrt{\frac{1}{m_0}} \|f\|_{H_0^\alpha}, \end{aligned}$$

where we have used the fact that $\|\cdot\|_{L^2} \leq C \|\cdot\|_{H_0^\alpha}$ for $\alpha > 0$, and that

$$\|g\|_{H_0^1} = \sqrt{2} \left\| G_{\frac{1}{2}}^{-1} g \right\|_{L^2} \leq 2 \sqrt{\frac{1}{m_0}},$$

by our choice of μ . The proof of Theorem 3.2 is thus completed. \square

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DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, UNIVERSITY OF ILLINOIS AT CHICAGO, 322 SCIENCE AND ENGINEERING OFFICES (M/C 249), 851 S. MORGAN STREET, CHICAGO, ILLINOIS 60607

E-mail address: `couyang@math.uic.edu`

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, UNIVERSITY OF ILLINOIS AT CHICAGO, 322 SCIENCE AND ENGINEERING OFFICES (M/C 249), 851 S. MORGAN STREET, CHICAGO, ILLINOIS 60607

E-mail address: `jpajda2@uic.edu`